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SUMMARY

A review of relativistic models of the universe and material relating to the cosmical constant showed that this term is a constant of integration and should be retained as a necessary term in the field equations of general relativity. An analysis of the zero-density model universe, a model universe that neglects density effects but retains the effect of curvature and the cosmical constant, showed that closed-form expressions could be obtained for the luminosity distance and redshift-magnitude relation. It was found that under certain conditions only small differences exist between the zero-density model universe and the finite-density model universe, which is physically close to the observed universe in composition but uniform in structure. It was concluded that under certain conditions the zero-density model universe offers a valuable tool for the analysis of observational data. The zero-density redshift-magnitude relation was used with observational data in a least-squares computing process to calculate new values of the acceleration parameter and the Hubble parameter. Because of the scarcity and quality of the data, the results of the calculations were not conclusive.

INTRODUCTION

The discovery of quasi-stellars (ref. 1) has extended earth-bound astronomical observations of cosmological interest to much larger redshifts. The introduction of orbiting astronomical observatories will permit the observation of galaxies and quasi-stellars at much larger redshifts and will, because of the opening of a wider band pass in the electromagnetic spectrum, enable the astronomer to obtain better apparent magnitudes for galaxies and quasi-stellars.

In model universes based on general relativity, the relationships used by the cosmologist to analyze observational data have, in the past, been approximate formulas or exact relationships for the special case of a zero cosmical constant. This paper examines the use of a zero-density model universe as a tool for the analysis of observational data. The relationships that connect theory and observation in a zero-density universe are exact. The strongest connection between theory and observation is the redshift-magnitude relation and this relationship for zero-density universes is studied in detail. It is shown that the redshift-magnitude relation for the zero-density universe is under certain conditions very useful for the analysis of observational data.

RELATIVISTIC MODELS OF THE UNIVERSE

Relativistic models of the universe are a natural branch from the theory of general relativity and have been reviewed in detail by Robertson (ref. 2) and in general by Bondi (ref. 3). The present discussion is confined to uniform models of the universe that have density greater than or equal to zero and pressure greater than zero. (See appendix for definition of symbols.) The metric for this type of universe is the Robertson-Walker metric and is

$$ds^2 = dt^2 - \frac{R^2(t)}{c^2}\frac{dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2}{\left(1 + \frac{kr^2}{4}\right)^2} \tag{1}$$

where ds has the dimensions of time along the line element, c the speed of light in vacuo, r, θ, ϕ dimensionless coordinates of a point, k the space curvature constant, and $R(t)$ is the scale factor that describes the manner in which space unfolds with time and has the dimensions of length. This metric (eq. (1)) was investigated independently by Walker (ref. 4) and Robertson (ref. 5) and was shown by them to hold for all uniform model universes. The relationship of the space time of experience, as given by equation (1), and the physical content of space time is provided by the 10 field equations of general relativity which are

$$G_{\mu\nu} - \frac{1}{2} G g_{\mu\nu} + \Lambda g_{\mu\nu} = -\kappa c^2 T_{\mu\nu} \tag{2}$$

where $G_{\mu\nu}$ is the Einstein-Ricci tensor and G its spur; $g_{\mu\nu}$ is the metrical tensor, κ is a constant and equal to $8\pi G/c^2$, Λ is the cosmological constant, and $T_{\mu\nu}$ is the energy tensor which is

$$T_{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) V_\mu V_\nu - \frac{p}{c^2} g_{\mu\nu} \tag{3}$$

for a single-stream isotropic fluid. In this equation, ρ and p are the density and pressure and V_μ the velocity is dx^μ/ds. All the terms in the field equations (eq. (2)) have now been defined except $\Lambda g_{\mu\nu}$ although Λ is the cosmical constant introduced by Einstein (ref. 6) in 1917 in a rather arbitrary manner. The retention or the elimination of this constant has been the subject of much controversy over the past decades. (See ref. 3.) The material presented in references 7 to 9 shows that the cosmical constant is actually a constant of integration and should be retained in the field equations. Consequently, this constant was retained in the field equations used in the studies reported in this paper.

The substitution of the space-time metric equation (eq. (1)) into the left-hand side of equation (2) and the energy tensor equation (eq. (3)) into the right-hand side of equation (2) by means of Dingle's formula (ref. 10) gives the well-known form of the Einstein field equations that define the cosmological problem for a homogeneous universe and are

$$\left.\begin{aligned} \frac{\dot{R}^2}{R^2} + \frac{2\ddot{R}}{R} + \kappa p &= -\frac{kc^2}{R^2} + \Lambda \\ \frac{\dot{R}^2}{R^2} - \frac{\kappa c^2 \rho}{3} &= -\frac{kc^2}{R^2} + \frac{\Lambda}{3} \end{aligned}\right\} \tag{4}$$

These differential equations give those densities and pressures that correspond to a given function $R(t)$ and the constants k and Λ. Only those sets of k, Λ, and $R(t)$ which give $\rho \geqq 0$, $p \geqq 0$ have physical significance and are of interest. The solution of these equations for $R(t)$ gives information concerning the geometrical history of the universe.

The previous discussion showed that Λ should be retained in the field equations; however, the physical meaning of Λ is vague. Substituting for $\dot{R}^2/R$ from the second into the first of equations (4) gives

$$\frac{\ddot{R}}{R} = \frac{\Lambda}{3} - \frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) \tag{5}$$

As can be seen from equation (5), Λ can be interpreted as an acceleration term. When Λ is negative, it corresponds to a force tending to slow down the expansion; when $\Lambda = 0$, only gravitational forces are acting; and when Λ is positive, it corresponds to a force tending to speed up the expansion.

Like Λ, k can also take on positive, negative, and zero values. In fact, these equations were derived in such a way that k may be defined as follows:

$$k = -1$$

$$k = 0$$

$$k = +1$$

For $k = -1$, space time is open and hyperbolic; for $k = 0$, space time is Euclidean; and for $k = 1$, space time is closed and spherical.

As shown by Robertson (ref. 2), equation (4) may be solved by quadrature when p and ρ are functions of time. Robertson analyzed these equations and obtained several families of model universes that are classified by the signs of k and Λ. Table I summarizes the results obtained by Robertson for those universes where energy is conserved and is greater than or equal to zero. The stationary universes have been omitted.

TABLE I.- FAMILIES OF FINITE-DENSITY MODEL UNIVERSES

Cosmical constant	Hyperbolic space; $k = -1$	Euclidean space; $k = 0$	Spherical space; $k = +1$
$\Lambda < 0$	Oscillating	Oscillating	Oscillating
$\Lambda = 0$	Expanding I	Expanding I	Oscillating
$\Lambda > 0$	Expanding I	Expanding I	Oscillating Expanding I Expanding II

The general form of the variation of $R(t)$ with time for each of these universes (expanding I, expanding II, and oscillating) is shown in figure 1.

The problem is to sort out from the 11 families of universes given in table I by using observational data, the family that best represents the actual (observed) universe. After this has been done, the member of the selected family which represents the actual universe can be determined. As indicated in table I, the signs of k and Λ determine the desired family of model universes, and since the equations have been adjusted so that k has the value ± 1 or 0, the magnitude of Λ specifies the exact model in the indicated family.

Equations (4) yield, without integration, interesting information concerning spatial curvature and the cosmical constant when they are evaluated for the present epoch. In making the evaluation for the present epoch, the pressure term can be considered negligible as it is about one-hundredth of the density term (ref. 11, p. 358). This assumption results in the zero-pressure universe. A zero subscript is used to denote the present epoch, and by dropping the pressure term, equations (4) become

$$\left.\begin{aligned} \frac{2\ddot{R}_o}{R_o} + \frac{\dot{R}_o^{\,2}}{R_o^{\,2}} &= -\frac{kc^2}{R_o^{\,2}} + \Lambda \\ \frac{\dot{R}_o^{\,2}}{R_o^{\,2}} - \frac{8\pi G\rho_o}{3} &= -\frac{kc^2}{R_o^{\,2}} + \frac{\Lambda}{3} \end{aligned}\right\} \tag{6}$$

According to accepted practice,

$$\frac{\dot{R}_o}{R_o} \equiv H_o \tag{7}$$

$$-\frac{\ddot{R}_o}{R_o H_o^{\,2}} \equiv q_o \tag{8}$$

$$\frac{4\pi G}{3H_o^2}\rho_o = \sigma_o \tag{9}$$

Here H_o is the Hubble parameter; q_o, the acceleration parameter; and σ_o, the density parameter. For a discussion of the density parameter see reference 12. The substitution of equations (7), (8), and (9) into equations (6) and solving for the Λ and kc^2/R_o^2 give

$$\Lambda = 3H_o^2(\sigma_o - q_o) \tag{10}$$

and

$$\frac{kc^2}{R_o^2} = H_o^2\left(3\sigma_o - q_o - 1\right) = 3\sigma_o H_o^2 + \dot{H}_o \tag{11}$$

where $\dot{H}_o = \left[\frac{d}{dt}\left(\frac{\dot{R}}{R}\right)\right]_{t=t_o}$ and is the present rate of change of the Hubble parameter. (See ref. 10, pp. 162 and 165.) By equation (11) H_o is a constant when $q_o = -1$; that is, when $\dot{H}_o = 0$ space is spherical and $k > 0$. However, there are spherical spaces in which H_o is not a constant and since the presently accepted value of q_o is greater than zero, then $\dot{H}_o$ exists in the observed universe. These results come from the field equations (eqs. (6)) when evaluated for the present epoch and no assumptions have been made with regard to $\dot{H}_o$.

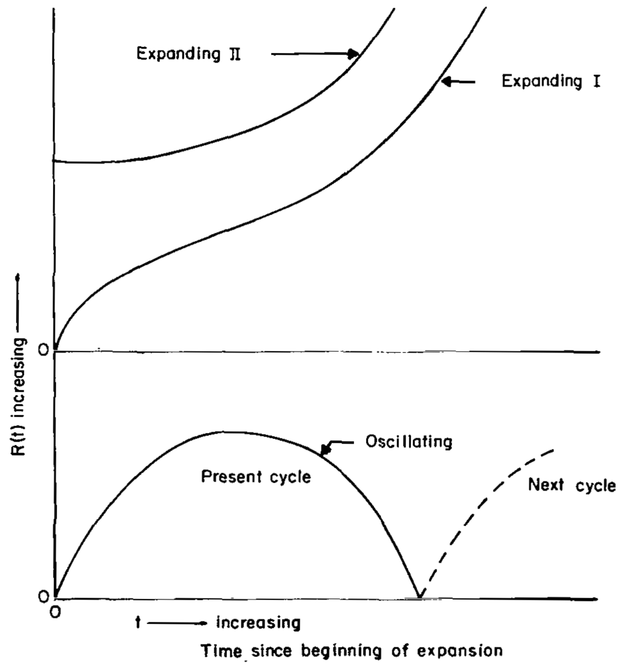

Figure 1.- Typical variations of R(t) with time for three basic families of models of universe. Origin for each family is taken as beginning of expansion.

If σ_o, q_o, and H_o can be determined by observation, then by equations (10) and (11), one of the families of model universes presented in table I is specified.

Some investigators make the assumption that $\Lambda = 0$ and claim that this assumption does not affect the generality of the results. If Λ is set equal to zero, equation (10) yields an expression for density which is

$$\rho_o = \frac{3H_o^2}{4\pi G} q_o \tag{12}$$

and equation (11) becomes

$$\frac{kc^2}{R_o^2} = H_o^2(2q_o - 1) = 3q_oH_o^2 + \dot{H}_o \tag{13}$$

This equation indicates that H_o will be a constant only when $q_o = -1$; thus, the space is hyperbolic. However, by equation (12) a value of q_o of -1 means negative density which is contrary to observational evidence. This result indicates that in realistic $\Lambda = 0$ models, H_o must always have a rate of change.

A study of equations (10) to (13) shows that the assumption $\Lambda = 0$ is highly restrictive. Setting $\Lambda = 0$ restricts the possible families of model universes to the oscillatory and expanding I type which originate in the singular state $R = 0$ at $t = 0$. Secondly, assuming $\Lambda = 0$ means that by equation (12), $q_o = \sigma_o$. This relation ties the acceleration parameter and the density parameter together. Such a relation should be established only by observational results and not by an arbitrary assumption. The assumption of $\Lambda = 0$ also rules out zero and negative values of q_o because these values would give negative and zero densities which cannot occur because of the existence of galaxies.

When Λ is not assumed to be zero, the density appears in the equations defining Λ and kc^2/R_o^2 as a parameter, that can be determined by observation, in defining the admissible sets of k and Λ for the determination of model universes. However, when Λ is assumed to be zero, the density (eq. (12)) is a function of q_o and H_o. If H_o is taken as 100 km/sec/Mparsec, ρ_o is $3.75 \times 10^{-29} q_o$, which, depending on the value of q_o, is 60 to 300 times the measured density of 3.1×10^{-31} g/cc. Even with an uncertainty factor of 10 which Oort (ref. 13) indicated exists for this measurement, the densities given by equation (12) are still large compared with the measured density. In order to have densities on the order of the measured ones, q_o must be small, between 0.01 and 0.1.

If σ_o and q_o are known, then equations (10) and (11) indicate the family of model universes that are of interest in the cosmological problem. If precise values are known for these parameters and for H_o, then a specific model of the family is specified. Current observational data indicate that σ_o lies between 0.015 and 1.5 and q_o is thought to lie between 0.5 and 2.5.

When these values are substituted into equations (10) and (11), Λ is found to be negative and k is equal to -1. This result indicates (see table I) that space is hyperbolic and the universe is oscillating. If Λ is arbitrarily set to zero, the current range of values for q_o indicates curvature constants of 0 or 1 and the universe is either an expanding I type and the space is Euclidean, or an oscillating model and the space is spherical. The results for $\Lambda = 0$ and $\Lambda \neq 0$ are very divergent when it is considered that the same values of q_o were used in each case.

RELATIONS BETWEEN THEORY AND OBSERVATION

The ability to differentiate between the families of model universes presented in table I and then between the members of the indicated family is dependent on the determination of H_o, q_o, and σ_o from observational data. For earth-based telescopes, the most important observables are: (1) apparent magnitude, (2) redshift, (3) numbers of galaxies, and (4) angular diameter.

As can be seen, there is not a direct connection between these parameters and those parameters of importance in cosmological theory. One of the tasks of the theorists is to construct a relation or series of relationships which connect theory and observation in terms of σ_o, q_o, and H_o. In doing this, the following relationships have been found to be useful:

(1) The redshift-magnitude relation

(2) The number-count relation

(3) The redshift angular-diameter relation

See reference 11 for examples of the use of these relationships. Both the redshift-magnitude relation and the count-magnitude relation depend on the luminosity distance, and the redshift angular-diameter relation depends on the cosmic distance (ref. 14) which differs from the luminosity distance by a factor of $(1 + \delta)^{-1}$, where δ is the redshift of the spectral lines due to the velocity of recession. Thus, once the expression for the luminosity distance has been determined, it is relatively easy to derive the subject relations. As the redshift-magnitude relation presents the strongest connection between observation and theory, and because of the ease with which the others may be derived from these distances if the methods of reference 10 or reference 15 are followed, the remainder of this paper will be primarily concerned with the redshift-magnitude relation.

The Redshift-Magnitude Relation

There are two forms of the redshift-magnitude relation in use for analysis of observational data. Both were derived for a uniform universe. The first of these relations, due to Robertson (ref. 14), is based on the assumption that the scale factor is sufficiently regular to be expanded about the conditions for the present epoch. This form of the redshift-magnitude relation is

$$m - K = 5 \log_{10} \frac{c\delta}{H_o} + 1.086\left(1 - q_o - 0.92 \frac{\dot{M}_o}{H_o}\right)\delta + M_o - 5 \tag{14}$$

For details of the derivation, see reference 10 or 14. In this expression m is the apparent magnitude; K, the correction for redshift; $\dot{M}_o$, the rate of change of absolute magnitude per epoch; M_o, the absolute magnitude; and 5 is a scale factor associated with the definition of absolute magnitude. In the form stated, equation (14) contains no effects from the curvature or cosmical constant. The retention of a δ^2 in equation (14) would introduce curvature for the present epoch but, in addition, the third time derivative of the scale factor for the present epoch would also appear in the coefficient of the δ^2 term. In light of present data it does not appear that this derivative could be evaluated. Lastly, this equation is restricted to small values of the redshift.

The second form of the redshift-magnitude relation was derived by Mattig (ref. 16) and is

$$m - K = 5 \log_{10} \frac{c}{H_o q_o^2}\left[q_o\delta + (q_o - 1)\left(\sqrt{1 + 2q_o\delta} - 1\right)\right] + M_o + \Delta M_o - 5 \tag{15}$$

In this equation ΔM_o corresponds to the term $1.086\left(-0.92 \frac{\dot{M}_o}{H_o} \delta\right)$ of equation (14) and is a correction for evolutionary effects in galaxies. Equation (15) is an exact redshift-magnitude relation for the case $\Lambda \equiv 0$ and a zero-pressure universe. Because it is an exact expression, equation (15) is not restricted to small δ and contains the effects of curvature and density. Since the condition that $\Lambda = 0$ was assumed in the derivation of the equation and since it was shown that $\Lambda \equiv 0$ is a highly restrictive assumption and that Λ is a necessary term in the field equations, equation (15) would not be useful in the analysis of data unless it can be shown from observational results that $\Lambda = 0$.

Derivation of a Redshift-Magnitude Relation for Arbitrary Λ

The present problem is to derive a redshift-magnitude relation which is valid for arbitrary values of Λ and for redshifts out to at least 1. The basic equation relating magnitudes and distance is

$$m - K = 5 \log_{10} D_l + M_o + \Delta M_o - 5 \tag{16}$$

where ΔM_o is used to represent the term $-1.086\left(0.92 \frac{\dot{M}_o}{H_o} \delta\right)$ of equation (14). Robertson (ref. 14) gives the general form of D_l as

$$D_l = R_o(1 + \delta)\ S(\omega) \tag{17}$$

where $S(\omega)$ is a function of the radial coordinate r of equation (1).

If $S(\omega)$ can be expressed as a function of H_o, q_o, σ_o, and δ, then the substitution of equation (17) into equation (16) gives the redshift-magnitude relation. The function $S(\omega)$ can be determined from the metric (eq. (1)). For the determination of $S(\omega)$, equation (1) is put in form (see ref. 10, ch. 8):

$$ds^2 = dt^2 - \frac{R^2}{c^2 R_o^2}\left[\frac{d\xi^2}{1 - \frac{k\xi^2}{R_o^2}} + \xi^2\left(d\theta^2 + \sin^2\theta\ d\phi^2\right)\right] \tag{18}$$

If the coordinates are chosen so that the origin is at the observer, then by spherical symmetry θ and ϕ are constants for any given light ray. The origin and references are so adjusted that $\theta = \phi = 0$. Light travels along a null geodesic and along a null geodesic $ds = 0$. Equation (18) can now be written as

$$d\omega = c\,\frac{dt}{R} = \frac{d\xi}{R_o\left(1 - \frac{k\xi^2}{R_o^2}\right)^{1/2}} \tag{19}$$

and

$$\omega = c\int_t^{t_o} \frac{dt}{R} = \int_0^{\xi} \frac{d\xi}{\left(R_o^2 - k\xi^2\right)^{1/2}} \tag{20}$$

which gives the connection between the time t and the radial coordinate ξ/R_o of the events observed from the origin at time t_o. Equation (20) gives two expressions for ω which are

$$\omega = \int_0^{\xi} \frac{d\xi}{\left(R_o^2 - k\xi^2\right)^{1/2}} \tag{20a}$$

and

$$\omega = c\int_t^{t_o} \frac{dt}{R} \tag{20b}$$

Equation (20a) can be integrated and the general solution is

$$\omega = \frac{1}{\sqrt{-k}} \sinh^{-1} \frac{\sqrt{-k}\xi}{R_o} \tag{21}$$

The inversion of equation (21) gives the function $S(\omega)$ and it is

$$S(\omega) = \frac{\xi}{R_o} = \frac{1}{\sqrt{-k}} \sinh \sqrt{-k}\omega \tag{22}$$

There are three specific solutions of interest for equation (22) that correspond to the three types of spatial curvature. The following table lists the specific forms of equation (22) as a function of the space-curvature constant:

k	Type of space	$S(\omega)$
-1	Hyperbolic	$\sinh \omega$
0	Euclidean	ω
+1	Spherical	$\sin \omega$

Equation (22) is indeterminant for $k = 0$. The specific solution can easily be obtained by expanding $\sinh \sqrt{-k}\omega$ and multiplying through by $1/\sqrt{-k}$ and then setting $k = 0$.

The integration of equation (20b) is accomplished in the form:

$$\omega = c \int_R^{R_o} \frac{dR}{\dot{R}R} \tag{23}$$

With the equation in this form, the second of equations (4) is used to eliminate $\dot{R}$ and equation (23) becomes

$$\omega = c \int_R^{R_o} \frac{dR}{R\sqrt{\frac{8\pi G\rho}{3} R^2 - kc^2 + \frac{\Lambda}{3} R^2}} \tag{24}$$

As a consequence of equation (4) specialized to the zero pressure universe, $\rho = \rho_o \frac{R_o^3}{R^3}$. This relation between ρ and ρ_o when combined with the definition of σ_o (eq. (9)) permits the term $\frac{8\pi G\rho}{3} R^2$ to be written as

$\frac{H_o{}^2\sigma_o R_o{}^3}{R}$. Through the use of equations (10) and (11) to eliminate kc^2 and Λ, the expression for ω finally becomes

$$\omega = c \int_R^{R_o} \frac{dR}{R^{1/2}\left[2H_o{}^2\sigma_o R_o{}^3 + H_o{}^2R_o{}^3(3\sigma_o - q_o - 1)R + H_o{}^2(\sigma_o - q_o)R^3\right]^{1/2}} \qquad (25)$$

The substitution of $R = \frac{R_o}{1 + \delta}$ into equation (25) expresses ω in terms of the redshift. The final expression for ω becomes

$$\omega = \frac{c}{H_oR_o} \int_0^{\delta} \frac{d\delta}{\left[2\sigma_o\delta^3 + (3\sigma_o + q_o + 1)\delta^2 + 2(q_o + 1)\delta + 1\right]^{1/2}} \qquad (26)$$

which is an elliptic integral that cannot be integrated by simple functions. The exact form of the redshift-magnitude relation for uniform zero-pressure model universes with finite density is given by equations (16), (17), and (22) and is

$$m - K = 5 \log_{10}\left[R_o(1 + \delta)\left(\frac{1}{\sqrt{-k}} \sinh \sqrt{-k}\omega\right)\right] + M_o + \Delta M_o - 5 \qquad (27)$$

where ω is given by equation (26).

This form of the redshift-magnitude relation is not easy to use for the analysis of observational data except when programed for use on a digital computer. In addition, little insight is obtained into what is happening when equation (27) is used on a digital computer. It is interesting to determine whether the integrand in equation (26) can be approximated by a quadratic, as was done by Mattig, without the use of highly restrictive assumptions such as $\Lambda = 0$ or that δ must be small compared with 1. If this could be done, equation (26) could then be integrated by simple functions and a simpler, more workable form of the redshift-magnitude relation would result.

One possible approach to the simplification of equation (26) is to neglect the terms involving σ_o. This approach is possible as σ_o is small and ranges from 0.015 to 0.15 and the terms $3\sigma_o\delta^2$ and $2\sigma_o\delta^3$ can be neglected with respect to $(q_o + 1)\delta^2$. The neglecting of σ_o constitutes an assumption of zero density. This approach to a simplification of the redshift-magnitude relation has been studied concurrently and independently by the author and G. C. McVittie of the University of Illinois.

The Zero-Density Model Universe

The idea of a zero-density universe was investigated by de Sitter in 1917. In 1932, Robertson (ref. 2) in his review of relativistic model universes discussed this and other zero density universes, that is, the $E = 0$ cases. In the past, zero-density universes were of interest because of the simplification of the mathematics introduced by this assumption. The simplifications permitted solutions of the equations describing the universe for special cases and were largely of academic interest. However, these solutions were useful in indicating trends and understanding a theory of the universe based on general relativity.

The assumption of zero density gives a model universe that is a limiting case of the finite-density model universe. In a zero-density model universe, matter has no effect on the underlying metric and curvature is solely a function of the geometry of space. Galaxies are test particles which have no effect on the structure and evolution of the universe. In the observable universe, galaxies and intergalactic matter give it a finite density; thus, the best representation would be a finite-density model. In this paper the differences between finite- and zero-density model universes are examined to determine whether the zero-density model universe is applicable to the reduction and analysis of observational data.

When ρ_o is assumed to be zero, σ_o goes to zero, and the equations for Λ and k (eqs. (10) and (11)) become

$$\Lambda = -3H_o^2 q_o \tag{28}$$

$$\frac{kc^2}{R_o^2} = -H_o^2\left(q_o + 1\right) = \dot{H}_o \tag{29}$$

Equation (29) shows that in the zero-density model universe, the Hubble parameter is a constant when $q_o = -1$ and space is Euclidean, $k = 0$. The type of space in which H_o can be a constant (see eq. (11)) and the condition that it be a constant in the zero-density model when space is Euclidean are two of the important differences between the finite and zero-density models of the universe.

Inasmuch as the signs of Λ and k define families of model universes, it is only when the signs of Λ and k given by equations (10) and (11) and (28) and (29) are the same that the zero-density model universe can be used in place of the finite-density model universe in the analysis of data. Table II compares k and Λ for the zero- and finite-density universes. A study of the data presented in table II shows that zero-density model universes may be used as analogues for the finite-density universes only when the following conditions occur in the finite-density model:

$$\left.\begin{array}{l} q_o > 0;\ \sigma_o < q_o;\ 3\sigma_o < (q_o + 1) \\ -1 < q_o < 0;\ 3\sigma_o - q_o < 1 \\ q_o < -1;\ \text{not restricted by tabulated information} \end{array}\right\} \tag{30}$$

All other cases listed in the table are excluded because the curvature constant and/or the cosmical constant do not have the same signs in the finite- and zero-density universes.

TABLE II.- ZERO- AND NONZERO-DENSITY MODEL UNIVERSES AS A FUNCTION OF q_o

Acceleration parameter	Density parameter			
	$\sigma_o \neq 0$ and $\Lambda = 3H_o^2(\sigma_o - q_o)$	$\sigma_o = 0$ and $\Lambda = -3H_o^2 q_o$	$\sigma_o \neq 0$ and $\frac{kc^2}{R_o^2} = H_o^2(3\sigma_o - q_o - 1)$	$\sigma_o = 0$ and $\frac{kc^2}{R_o^2} = -H_o^2(q_o + 1)$
$q_o > 0$	$\sigma_o > q_o;\ \Lambda > 0$ $\sigma_o = q_o;\ \Lambda = 0$ $\sigma_o < q_o;\ \Lambda < 0$	$\Lambda < 0$	$3\sigma_o > (q_o + 1);\ k = 1$ $3\sigma_o = q_o + 1;\ k = 0$ $3\sigma_o < (q_o + 1);\ k = -1$	$k = -1$
$q_o = 0$	$\Lambda > 0$	$\Lambda = 0$	$3\sigma_o > 1;\ k = 1$ $3\sigma_o = 1;\ k = 0$ $3\sigma_o < 1;\ k = -1$	$k = -1$
$-1 < q_o < 0$	$\Lambda > 0$	$\Lambda > 0$	$3\sigma_o - q_o > 1;\ k = 1$ $3\sigma_o - q_o = 1;\ k = 0$ $3\sigma_o - q_o < 1;\ k = -1$	$k = -1$
$q_o = -1$	$\Lambda > 0$	$\Lambda > 0$	$k = 1$	$k = 0$
$q_o < -1$	$\Lambda > 0$	$\Lambda > 0$	$k = 1$	$k = 1$

When conditions exist for the use of a zero-density model of the universe, differences will occur in Λ and R_o for the finite- and zero-density model universes. Table III compares Λ, k, and R_o for finite- and zero-density universes.

The differences in Λ range from approximately 3 percent to more than 18 percent. A larger difference occurred for $q_o = 0.5$ and $\sigma_o = 0.8$; however, in this case the condition $3\sigma_o < q_o + 1$ (eq. (30)) for the use of a zero-density model had been violated. The differences in R_o varied from about 4 percent to more than 43 percent and the inadmissible case of $q_o = 0.5$, $\sigma_o = 0.8$ shows a difference in R_o of about 22 percent. These results indicate a high sensitivity of both Λ and R_o to the assumption of zero density. As both Λ and R_o appear in the equations for the luminosity distance and redshift-magnitude relation, a careful study is required to determine whether the differences in Λ and R_o introduce unacceptably large differences in the luminosity distance and redshift-magnitude relation.

The results presented in table III also show the importance of the restrictions given by the inequalities (eq. (30)) for the use of zero-density model universes. When $q_o = 0.5$ and $\sigma_o = 0.8$, the first of the inequalities (eq. (30)) was not satisfied and both k and Λ change sign; therefore, a zero-density universe cannot be used to analyze a finite-density universe when $q_o = 0.5$ and $\sigma_o = 0.8$. However, when $q_o = 2.5$ and $\sigma_o = 0.8$, the

TABLE III.- Λ, k, AND R_o FOR ZERO- AND FINITE-DENSITY MODEL UNIVERSES

q_o	k	Λ for σ_o of -			$\frac{\Lambda_{\sigma_o \neq 0} - \Lambda_{\sigma_o = 0}}{\Lambda_{\sigma_o = 0}}$, percent, for σ_o of -		R_o for σ_o of -			$\frac{R_{o,\sigma_o \neq 0} - R_{o,\sigma_o = 0}}{R_{o,\sigma_o = 0}}$, percent, for σ_o of -	
		0	0.08	0.8	0.08	0.8	0	0.08	0.8	0.08	0.8
0.5	-1	-1.58×10^{-35}	-1.33	-----	18.80	-----	7.56×10^{27}	8.24	----	8.24	-----
	1	------------	-----	1.05	-----	50.25	----------	----	9.75	----	22.45
2.5	-1	-7.89	-7.64	-5.37	3.27	9.86	4.95	5.13	8.72	3.52	43.50
	1	------------	-----	-----	-----	-----	----------	----	----	----	-----

inequality was satisfied and the signs of k and Λ remained the same as those of the finite-density model; thus, the zero-density model could be used to analyze observational data for these conditions.

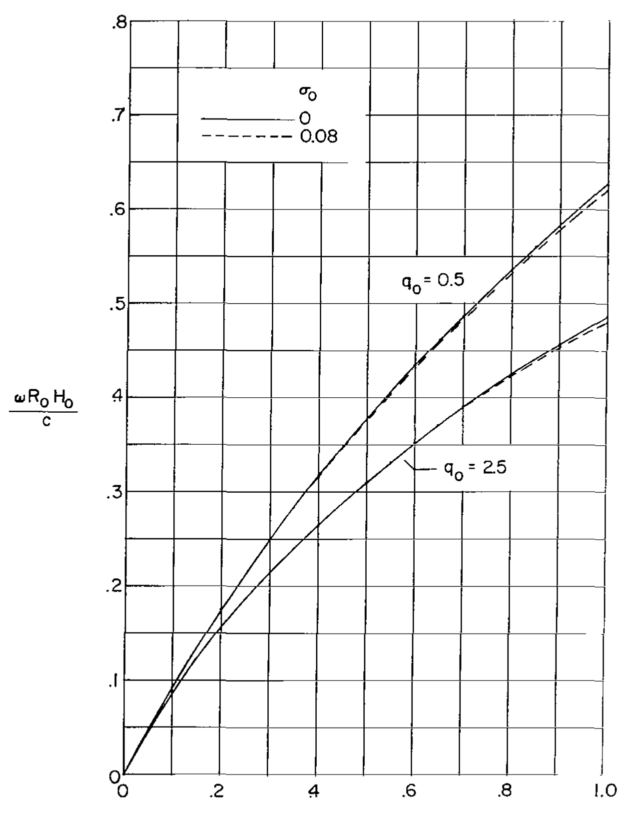

Figure 2.- Differences in $\omega R_o H_o/c$ between finite-density and zero-density model universes.

For the analysis of the zero-density model universe, equation (26) was written in the form

$$\omega = \frac{c}{H_o R_o} \int_0^{\delta} \frac{d\delta}{\sqrt{\frac{-kc^2}{H_o^2 R_o^2}(1 + \delta)^2 - q_o}} \tag{31}$$

by setting $\sigma_o = 0$ and using equation (28) to eliminate Λ. Figure 2 compares $\omega R_o H_o/c$ for the finite-density universe (eq. (26)) and that for the zero-density universe (eq. (31)) for $q_o = 0.5$ and 2.5 and $\sigma_o = 0$ and 0.08. The differences between zero- and finite-density universes are small over the range of δ considered. For $q_o = 0.5$, differences between the two solutions start at $\delta = 0.4$, and for $q_o = 2.5$, the differences start at $\delta = 0.65$. When δ becomes greater than 1, the density terms which are functions of δ^3 and δ^2 start to increase and the differences between the zero- and finite-density models would become larger.

The good agreement obtained for $\omega R_o H_o / c$ for the finite- and zero-density model universes indicates that this quantity is relatively insensitive to σ_o between $\delta = 0$ and $\delta = 1$. Systematic calculations showed that for $q_o > 0$ and $0 \leqq \delta \leqq 1$, differences of 2 percent or less in $\omega R_o H_o / c$ would occur if

$$\sigma_o \leqq 0.075 q_o (q_o - 1) + 0.1 \tag{32}$$

The next step in obtaining an expression for the luminosity distance in the zero-density model universe is to integrate equation (31). The expression $\zeta = \frac{\sqrt{-kc}}{H_o R_o}(1 + \delta)$ was substituted into this equation to obtain

$$\omega = \frac{1}{\sqrt{-k}} \int_{\frac{\sqrt{-kc}}{H_o R_o}}^{\frac{\sqrt{-kc}}{H_o R_o}(1+\delta)} \frac{d\zeta}{\sqrt{\zeta^2 - q_o}} \tag{33}$$

which integrates as a log function. Subsequent integration yields

$$\omega = \frac{1}{\sqrt{-k}} \left[\log \frac{\frac{\sqrt{-kc}}{H_o R_o}(1 + \delta) + \sqrt{\frac{-kc}{H_o^2 R_o^2}(1 + \delta)^2 - q_o}}{\frac{\sqrt{-kc}}{H_o R_o} + \sqrt{\frac{-kc^2}{H_o^2 R_o^2} - q_o}} \right]$$

which is the general solution for ω. This solution is indeterminate for $q_o = -1$. The correct specific solution for $q_o = -1$ is easily obtained by expanding the logarithm and then multiplying the expansion by $\frac{1}{\sqrt{-k}}$ and letting k approach zero. A more convenient form of the general solution for ω is the inverse hyperbolic cosine representation which is

$$\omega = \frac{1}{\sqrt{-k}} \left[\cosh^{-1} \frac{\sqrt{-kc}(1 + \delta)}{H_o R_o \sqrt{q_o}} - \cosh^{-1} \frac{\sqrt{-kc}}{H_o R_o \sqrt{q_o}} \right] \tag{34}$$

Figure 3 compares ω for $q_o > 0$, $k = -1$ as given by equation (26) for the finite-density case and by equation (34) for the zero-density case. For $q_o > 0$ and $k = -1$, the specific expression for ω when $\rho_o = 0$ is

$$\omega = \cosh^{-1} \sqrt{\frac{q_o + 1}{q_o}}(1 + \delta) - \cosh^{-1} \sqrt{\frac{q_o + 1}{q_o}} \tag{35}$$

and this is the equation actually plotted in figure 4. In figure 4 values of q_o of 0.5 and 2.5 and values of σ_o of 0 and 0.8 were used. Unlike the

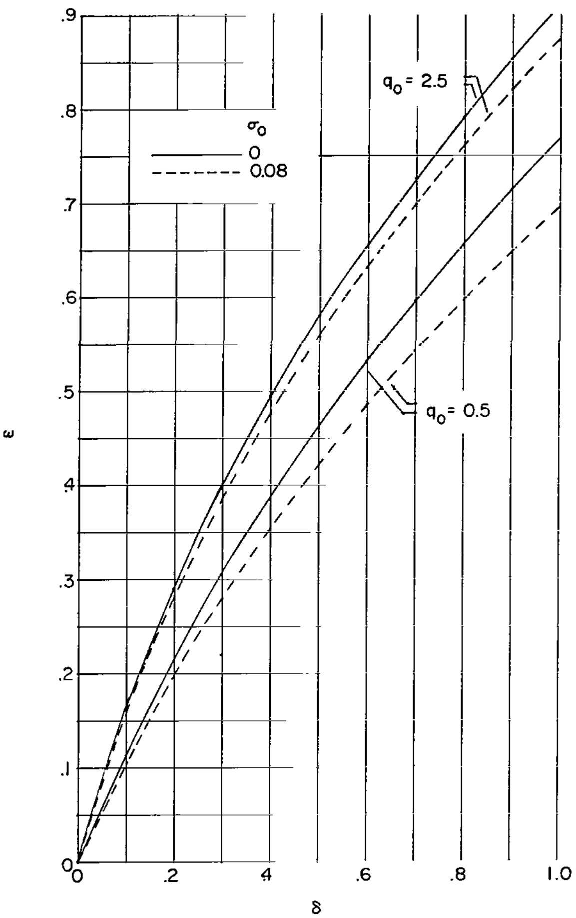

Figure 3.- Differences in ω for finite-density and zero-density model universes.

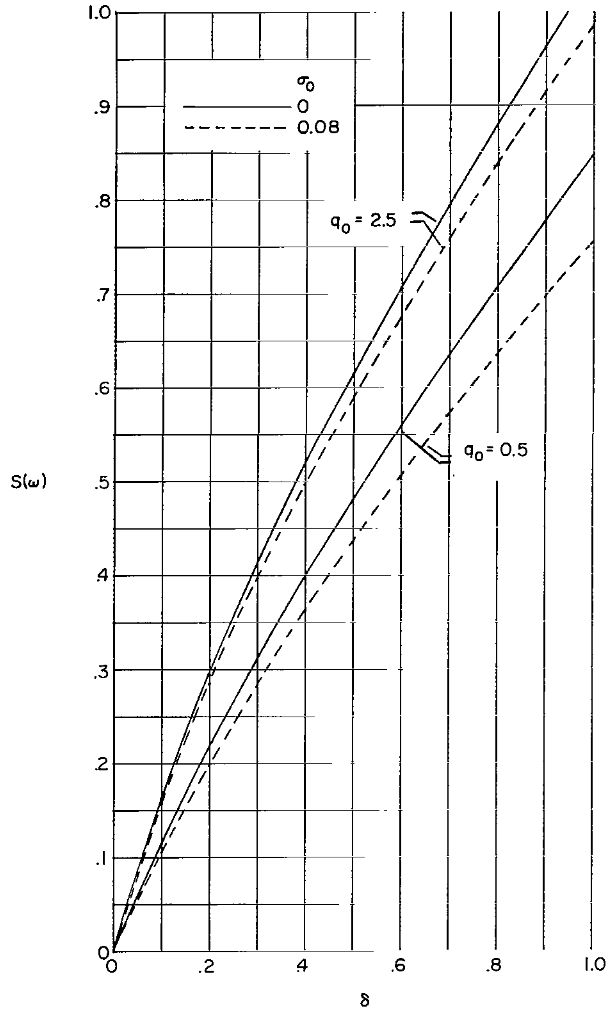

Figure 4.- Differences in $S(\omega)$ for finite-density and zero-density model universes.

previous case large differences start to show up almost immediately. These larger differences are caused by the differences in R_O for the zero- and finite-density universes. (See table III.) Two nearly equal quantities are being divided by different R_O values and larger differences result.

Substitution of equation (34) into equation (22) gives the general solution for $S(\omega)$ which is

$$S(\omega) = \frac{1}{\sqrt{-k}} \sinh \sqrt{-k}\left\{\frac{1}{\sqrt{-k}}\left[\cosh^{-1} \frac{\sqrt{-k}c(1 + \delta)}{H_o R_o \sqrt{q_o}} - \cosh^{-1} \frac{\sqrt{-k}c}{H_o R_o \sqrt{q_o}}\right]\right\} \tag{36}$$

After some manipulation, equation (36) reduces to

$$S(\omega) = \frac{c}{H_o R_o q_o}\left[\sqrt{(q_o + 1)(1 + \delta)^2 - q_o} - (1 + \delta)\right] \tag{37}$$

for the general solution of $S(\omega)$. In the case $q_o = 0$, equation (37) becomes indeterminant. One differentiation of the numerator and denominator with respect to q_o produced the solution for $q_o = 0$. Figure 4 compares $S(\omega)$ for finite-density universes; $S(\omega)$ is obtained by substituting equation (26) into equation (22) for the finite-density case, and is given by equation (37) for the zero-density case. The differences shown in figure 4 are on the order of those shown in figure 3 for the variable ω. The results for ω and $S(\omega)$ show that large differences exist between the finite- and zero-density universes. These differences are large enough so that the use of a zero-density model universe for the analysis of observational data is questionable on the basis of the comparison with the finite-density universe up to this point. Fortunately, $S(\omega)$ does not enter directly into the analysis of observational data. The function $S(\omega)$ is used to obtain the luminosity distance from equation (17) in terms of the redshift and the luminosity distance is the critical parameter for the analysis of observational data.

The luminosity distance is obtained by substituting equation (37) into equation (17). This substitution gives

$$D_l = \frac{c(1 + \delta)}{H_o q_o}\left[\sqrt{(q_o + 1)(1 + \delta)^2 - q_o} - (1 + \delta)\right] \tag{38}$$

for the luminosity distance in the zero-density universe. This is a general solution for D_l and like $S(\omega)$ is indeterminate for $q_o = 0$. However, the correct specific solution is obtained by differentiating the numerator and denominator once with respect to q_o and then evaluating D_l for $q_o = 0$.

The cosmic distance (ref. 14) which is used in the analysis of angular and isophotal diameters can be obtained from equation (38) by dividing by $1 + \delta$.

The luminosity distance for the finite-density universe is obtained by the substitution of equations (22) and (26) in that order into equation (17). Figure 5 compares the luminosity distances for the finite- and zero-density model universes, the latter being given by equation (38). The differences between D_l for the finite- and zero-density model universes are small and good agreement is obtained between the finite- and zero-density model universes out to a δ of about 0.6 at $q_o = 0.5$ and 0.75 for $q_o = 2.5$. This agreement is somewhat fortuitous because large differences introduced when $\omega H_o R_o/c$ was multiplied by $c/H_o R_o$ to obtain ω have been eliminated by the multiplication by R_o when D_l was obtained from equations (17) and (37). The good agreement obtained for D_l for the finite- and zero-density model universes

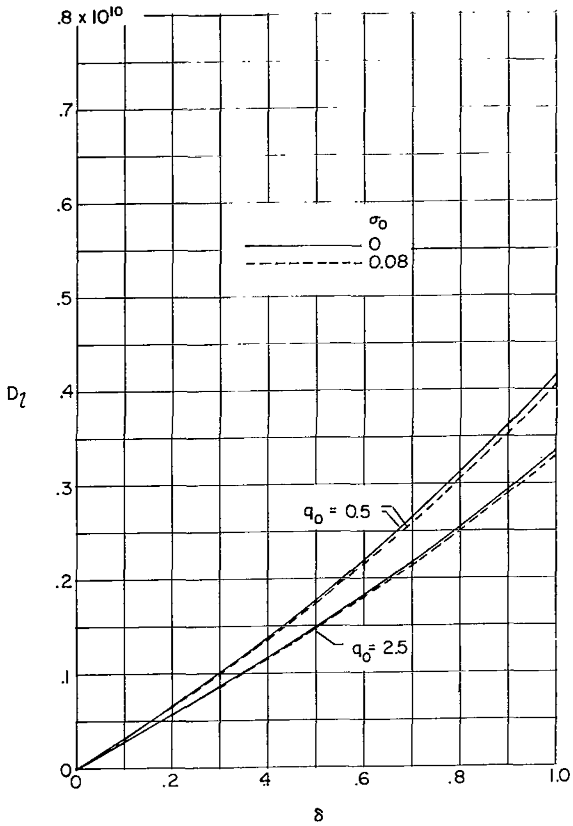

Figure 5.- Comparison of luminosity distances for finite-density and zero-density model universes.

indicates that D_l as given by the zero-density model can be used for the analysis of observational data.

The radical in the expression for D_l (eq. (38)) indicates that there may be regions where the luminosity distance becomes complex. For any value of q_o the value of δ at which the radical becomes zero can be found by setting the radicand in equation (38) equal to zero and solving for δ. The critical value of δ, called δ_c, is given by

$$\delta_c = \sqrt{\frac{q_o}{q_o + 1}} - 1 \tag{39}$$

and is the boundary between real and imaginary square roots in equation (39); thus, it is also the boundary between real and complex luminosity distances. In an expanding universe where $\delta > 0$, equation (39) imposes no restrictions for $q_o > 0$, $q_o = 0$, $0 > q_o > -1$, and $q_o = -1$ as the critical value of δ is either negative or complex in these regions. However, for $q_o < -1$ the radical in equation (39) is positive and greater than one. This condition means that in the region $q_o < -1$, there are values of δ greater than zero at which the luminosity distance becomes complex; for example, for $q_o = 2.5$, $\delta_c = 0.2955$. For δ greater than 0.2955 the luminosity distance will be complex and for δ less than 0.2955 the luminosity distance will be real. These results indicate that for $q_o < -1$,

$$\delta \leqq \sqrt{\frac{q_o}{q_o + 1}} - 1 \tag{40}$$

and equation (40) is a restriction that must be added to the restrictive conditions (eq. (30)).

The redshift-magnitude relation is obtained by substituting equation (38) into equation (16). The redshift-magnitude relation for the zero-density model universe is

$$m - K = 5 \log_{10}\left\{\frac{c(1 + \delta)}{H_o q_o}\left[\sqrt{(q_o + 1)(1 + \delta)^2 - q_o} - (1 + \delta)\right]\right\} + M_o + \Delta M_o - 5 \tag{41}$$

This redshift-magnitude relation is good for all q_o provided the argument of the log term is handled as an indeterminant form when $q_o = 0$. Figure 6 compares redshift-magnitude relations for the finite- and zero-density universes. In these calculations $q_o = 1.5$, $H_o = 100$ km/sec/Mparsec, $M_o = -20.56$, and $\Delta M_o = 0$. Four distinct models are shown; they are $\sigma_o = 0$, 0.08, 0.16, and 0.8. The curves for $\sigma_o = 0$, 0.08, and 0.16 fall almost on top of each other. There are very small differences between these three that are not discernible in figure 6. With respect to the observable universe, a value of σ_o of 0.08 corresponds to a density about half way between $\rho_o = 3.1 \times 10^{-31}$ and $\rho_o = 3.1 \times 10^{-30}$ gram per cubic centimeter which is the density range given by Oort in reference 13. The model with $\sigma_o = 0.16$ roughly corresponds to the upper limit of Oort's range of density. When $\sigma_o = 0.8$, large differences occurred and this value of σ_o corresponds to a density of approximately 1.6×10^{-29} g/cc which is about five times the upper limit of the density range given by Oort. Even at $\sigma_o = 0.8$, agreement between the zero- and finite-density models is very good out to $\log c\delta = 4.8$ which corresponds to a δ of about 0.21. The inequality (eq. (33)) indicates, as a function of q_o, a value of σ_o that will give a difference of 2 percent or less between the finite- and zero-density model universes at $\delta = 1$. For the value of q_o used

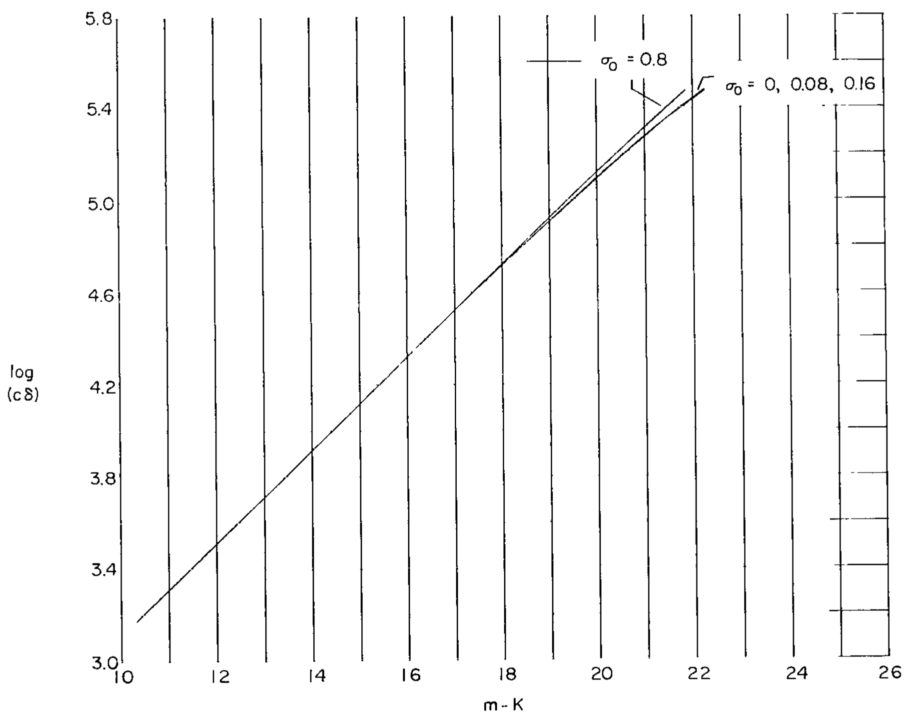

Figure 6.- Comparison of redshift-magnitude relation for finite-density and zero-density model universes. $H_o = 100$ km/sec/Mparsec; $q_o = 1.5$; $M_o = -20.56$; and $\Delta M_o = 0$.

in preparing figure 6, the inequality (eq. (33)) indicates that for $\sigma_o \approx 0.16$, the difference between the finite- and zero-density model universes will be 2 percent or less. On the plot little or no differences occur between $\sigma_o = 0$ and $\sigma_o = 0.16$ models; at $\sigma_o = 0.8$, large differences occur and σ_o is five times that given by equation (33).

The results obtained in this section show that a zero-density model universe is under certain conditions a good analogue of a finite-density model universe and can be used to analyze a universe with finite density. As the observable universe is a finite-density universe, the zero-density model universe can be used for the analysis of observational data when the following conditions (see table IV) have been met in the observed universe.

TABLE IV.- CONDITIONS FOR USE OF ZERO-DENSITY MODEL UNIVERSE

FOR THE ANALYSIS OF OBSERVATIONAL DATA

Condition		Results
General:		
$\sigma_o \leqq 0.075q_o(q_o - 1) + 0.1$		Errors of less than 2 percent at $\delta = 1$
Specific:		
$q_o > 0$	$q_o > \sigma_o$ $q_o + 1 > 3\sigma_o$	k and Λ of the observable and zero-density universes have same sign
$0 > q_o > -1$	$1 > 3\sigma_o - q_o$	
$q_o < -1$	$\delta \leqq \sqrt{\frac{q_o}{q_o + 1}} - 1$	Eliminates complex luminosity distances

One use of the redshift-magnitude relation is the reduction of observational data in a least-squares computing process. In this type of computation, the constants in the equations are regarded as undetermined. The least-squares process then seeks to determine the best values of these constants from observational data. In the exact form of the redshift-magnitude relation, there are three such constants, q_o, H_o, and σ_o. However, in the zero-density redshift-magnitude relation (eq. (41)) only two constants, H_o and q_o, are available. How the values of H_o and q_o are affected by this third undetermined constant in fitting observational data must be determined. Most important, the loss of σ_o as a curve-fitting parameter prevents a better determination of the density of the universe, inasmuch as the effect of density is contained in the observational data.

In model universes the second of equations (4) can be integrated to obtain $t_o - t$. The inversion of the solution for $t_o - t$ gives the solution for the scale factor R. In the zero-density model universe the second of equations (4) is

$$\frac{\dot{R}^2}{R^2} = -\frac{kc^2}{R^2} + \frac{\Lambda}{3} \tag{42}$$

This equation when solved for dt and when equations (28) and (29) are used to eliminate Λ and k becomes

$$dt = \frac{dR}{H_o\left[R_o^2(q_o + 1) - q_oR^2\right]^{1/2}} \tag{43}$$

and

$$H_o(t_o - t) = \int_R^{R_o} \frac{dR}{\left[R_o^2(q_o + 1) - q_oR^2\right]^{1/2}} \tag{44}$$

The integration of equation (44) gives the general expression for $t_o - t$ which is

$$H_o(t_o - t) = \frac{1}{\sqrt{q_o}}\left(\cos^{-1} \frac{R}{R_o}\sqrt{\frac{q_o}{q_o + 1}} - \cos^{-1}\sqrt{\frac{q_o}{q_o + 1}}\right) \tag{45}$$

Equation (45) which gives $t_o - t$ for the zero-density universe is compared with $t_o - t$ for a finite-density universe with $\sigma_o = 0.08$ in figure 7. When $q_o = 0.5$, the two solutions showed slight differences above $\delta = 0.65$. For $q_o = 2.5$, no difference is detectable in figure 7. The inversion of equation (45) gives the expression for the scale factor which is

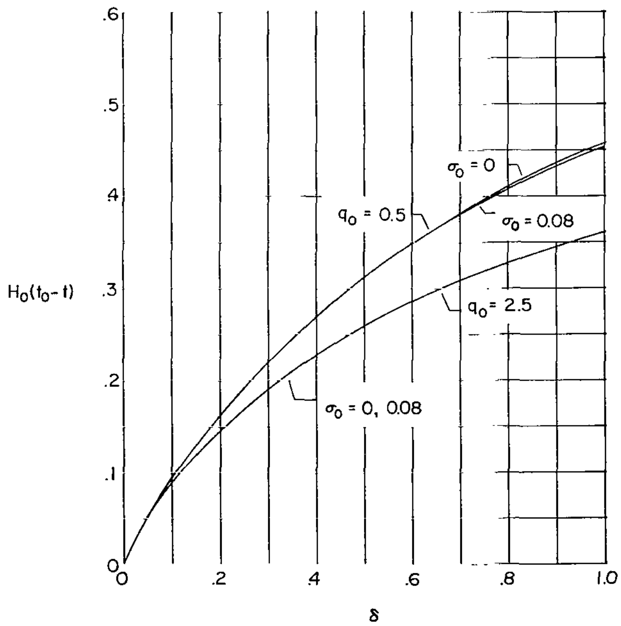

Figure 7.- Comparison of $H_o(t_o - t)$ for finite-density and zero-density model universes.

$$R = \frac{c}{H_o\sqrt{q_o + 1}}\left[\cos \frac{\sqrt{q_o}(t_o - t)}{H_o^{-1}} - \frac{1}{\sqrt{q_o}} \sin \frac{\sqrt{q_o}(t_o - t)}{H_o^{-1}}\right] \tag{46}$$

Figure 8 compares H_oR for finite-density and zero-density universes for $q_o > \sigma_o > 0$. The density parameter for the finite-density universe was taken to be equal to 0.08. Two values of q_o, 0.5 and 2.5, were considered; in both cases the lack of good agreement, such as obtained for $\omega H_oR_o/c$ or D_l, is most noticeable. At $q_o = 2.5$, the differences are smaller than at $q_o = 0.5$ because at large q_o the assumption of zero density produces a smaller effect. When compared with figure 7, the differences shown in figure 8 must be considered large. In equation (46) the term before the brackets is $c/H_o\sqrt{q_o + 1}$ and is, by equation (29), R_o. It was previously shown (see table III and the associated discussion) that R_o is very sensitive to σ_o and large differences occur between the $\sigma_o = 0$ and $\sigma_o \neq 0$ values of R_o when H_o and q_o are held constant. The differences shown in figure 8 reflect these differences in R_o. The differences do not occur in figure 7 because the ratio R/R_o appears in equation (45) rather than R_o by itself. The ratio R/R_o is equal to $(1 + \delta)^{-1}$ which is the same in both finite- and zero-density universes.

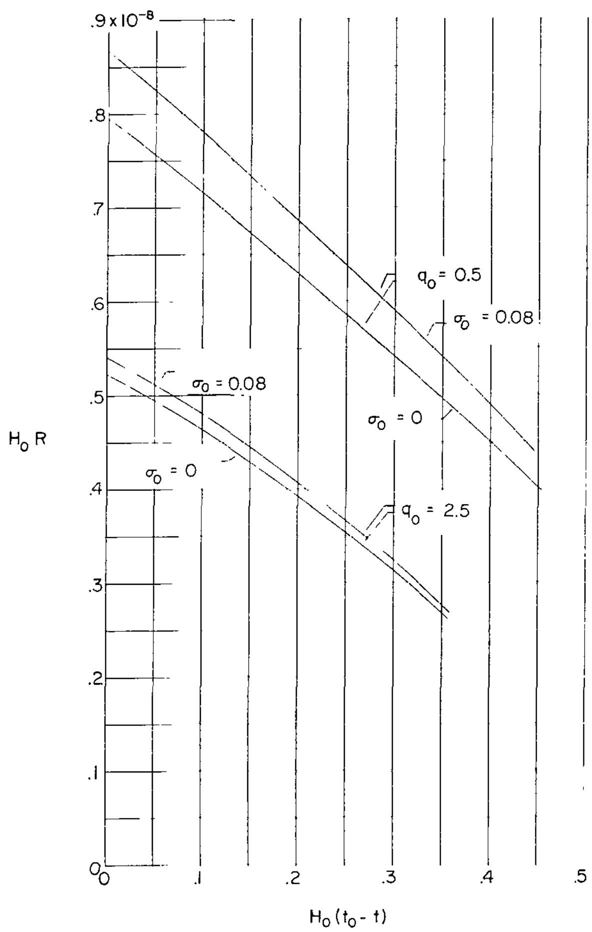

Figure 8.- Comparison of scale factor R for finite-density and zero-density model universes.

Extensive calculations with the digital computer showed that inequality (eq. (33)) when met insures that the differences between the zero- and finite-density model universes will be 2 percent or less at $\delta = 1$ when referred to the finite-density solutions.

These results show that for the analysis and reduction of observational data, the zero-density model can be used for time analysis. However, because of the large differences that occur in R between the finite- and zero-density models, the use of a zero-density model to determine a scale factor for the observable universe appears at best to be highly questionable.

There is one other relation connecting theory and observation that may become important. This is the previously mentioned count-magnitude relation (see ref. 10). This relationship has not been too useful because the number of galaxies observed at great distances was not sufficient to give a significant distribution. The use of large telescopes in orbit around the earth to record distant galaxies may produce sufficient information so that this relation would give significant correlation between observation and theory.

The count-magnitude relation is

$$N(m) = \frac{nR_0^3}{Q} \int_0^{\omega} \int_0^{\pi} \int_0^{2\pi} S^2(\omega)\, d\omega$$

where $N(m)$ is the number of galaxies equal to or brighter than a given magnitude, n the number of galaxies per unit volume, and Q is the number of square degrees in the celestial sphere. For the zero-density model this expression integrates to

$$N(m) = \frac{2\pi nR_0^3}{(-k)Q}\left(\sinh \sqrt{-k}\omega \cosh \sqrt{-k}\omega - \sqrt{-k}\omega\right)$$

A study of this expression, based on the analysis of ω and $S(\omega)$, indicates that there would be considerable differences between the count-magnitude relation for $\rho_0 = 0$ and for $\rho_0 \neq 0$, the more physically correct finite-density model universe. For this reason a count-magnitude relation for the zero-density universe has not been reported in detail.

Calculations With Redshift-Magnitude Relation

In the previous section it was shown that under certain conditions the redshift-magnitude relation for the zero-density universe can be used for the analysis of observational data. The redshift-magnitude relation (eq. (41)) is compared with other redshift-magnitude relations and the sensitivity to q_0 is investigated.

Figure 9 shows the effects of q_0 on the redshift-magnitude relation (eq. (41)), that is, the separation of the curves for different q_0. This separation is important inasmuch as the magnitude of separation will indicate how far observations must be extended in order to obtain data so that a model of the universe will be defined without question. The separation of the curves for different q_0 is so poor out to a redshift of 1 that differences are hard to judge on the figure. Analysis of the data showed that at $\delta = 0.46$, the present limit of observation for galaxies, the curves for $q_0 = 0.5$ and $q_0 = 1.0$ were separated by 0.11 magnitude and the curves for $q_0 = 2.0$ and $q_0 = 2.5$ by 0.08 magnitude. At $\delta = 1$ the differences were 0.15 magnitude and 0.09 magnitude, respectively. (Quasi-stellars with confirmed redshifts of

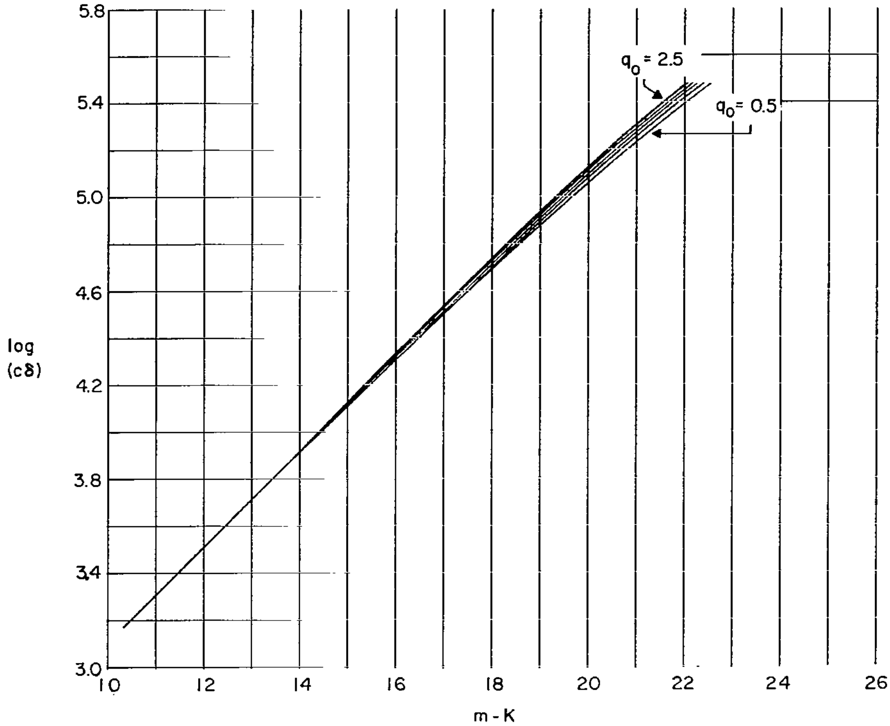

Figure 9.- Effect of varying q_o on zero-density redshift-magnitude relation (eq. (41)). q_o taken in steps of 0.5; H_o = 100 km/sec/Mparsec; M_o = -20.56; and ΔM_o = 0.

0.545 have been observed and are reported in ref. 17.) It can be concluded that if this form of the redshift-magnitude relation represents the relation between the redshift and apparent magnitude in the observed universe, then observations must be extended to redshifts considerably larger than one before a model of the universe can be established.

The next step is to learn whether the zero-density redshift-magnitude relation (eq. (41)) is a significant improvement over other such relationships that are in use. Figure 10 compares the zero-density redshift-magnitude relation with the approximate redshift-magnitude relation (eq. (14)) when $\dot{M}_o = 0$. The difference in magnitude between these equations Δm was obtained by subtracting equation (14) from equation (41).

$$\Delta m = 5 \log \left\{ \frac{1 + \delta}{q_o \delta} \left[\sqrt{(q_o + 1)(1 + \delta)^2 - q_o} - (1 + \delta) \right] \right\} - 1.086(1 - q_o)\delta$$

The differences range from about 0.005 magnitude at $\delta = 0.1$, $q_o = 0.5$ to 1.86 magnitudes at $\delta = 1$, $q_o = 2.5$. The difference at $\delta = 0.46$, the redshift of the galaxy 3C295, ranges from 0.065 magnitude at $q_o = 0.5$ to 0.7 magnitude at $q_o = 2.5$. Equation (41) was expanded to see whether the expansion would show where the differences were arising. The expanded form of equation (41) is

$$m - K = 5 \log_{10} \frac{c\delta}{H_o} + 1.086(1 - q_o)\delta + 0.2715\left(3q_o^{\,2} + 6q_o - 1\right)\delta^2 + M_o + \Delta M_o - 5 \tag{47}$$

subject to the restriction that $\left|(1 - q_o)\frac{\delta}{2} + q_o(q_o + 1)\frac{\delta^2}{2}\right| < 1$. Comparison with equation (14) shows that the differences occur in δ^2, the higher order terms that have been neglected in equation (47). Since curvature and the effect of cosmical constance do not appear until the δ^2 and higher order terms are retained in the expansion, the differences shown in figure 10 are a function of these parameters.

A similar comparison was made between equation (41) and equation (15), Mattig's form of the redshift-magnitude relation for $\Lambda = 0$, and was used by Sandage (ref. 11) to analyze observational results. The difference Δm shown in figure 11 was obtained by subtracting equation (15) from equation (41).

$$\Delta m = 5 \log\left\{\frac{1+\delta}{q_o}\left[\sqrt{(1+q_o)(1+\delta)^2 - q_o} - (1+\delta)\right]\right\} - 5 \log\left\{\frac{1}{q_o^{\,2}}\left[q_o\delta + (q_o - 1)\left(\sqrt{1 + 2q_o\delta} - 1\right)\right]\right\}$$

In this case Δm ranges from about 0.01 magnitude at $\delta = 0.1$, $q_o = 0.5$ to 0.865 magnitude at $\delta = 1.0$, $q_o = 2.5$. The expanded form of equation (15) is

$$m - K = 5 \log_{10} \frac{c\delta}{H_o} + 1.086(1 - q_o)\delta + 0.2715\left(3q_o + 1\right)\left(q_o - 1\right)\delta^2 + M_o + \Delta M_o - 5 \tag{48}$$

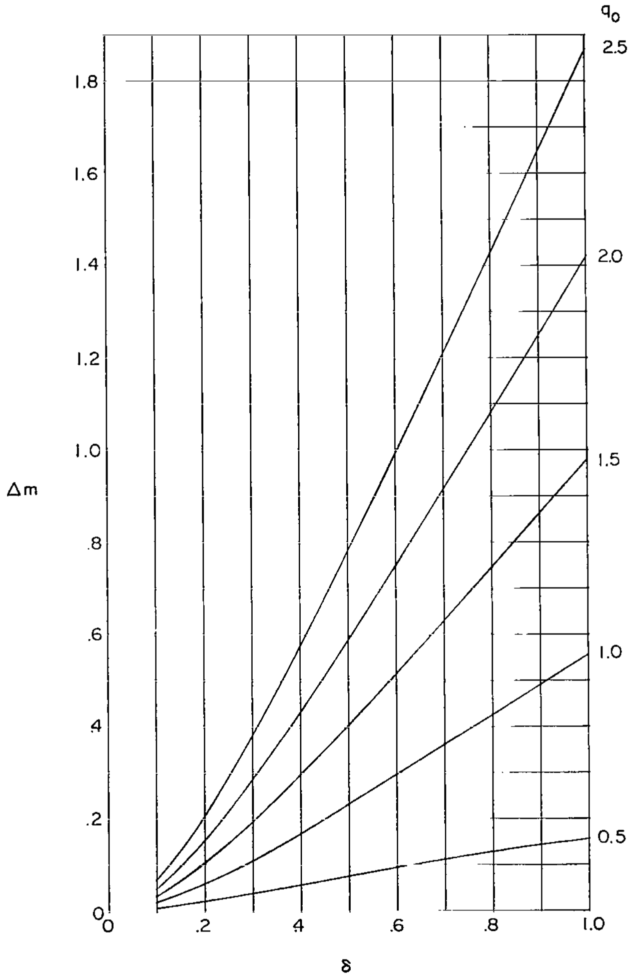

Figure 10.- Difference, as an incremental magnitude, introduced by use of equation (14) instead of equation (41).

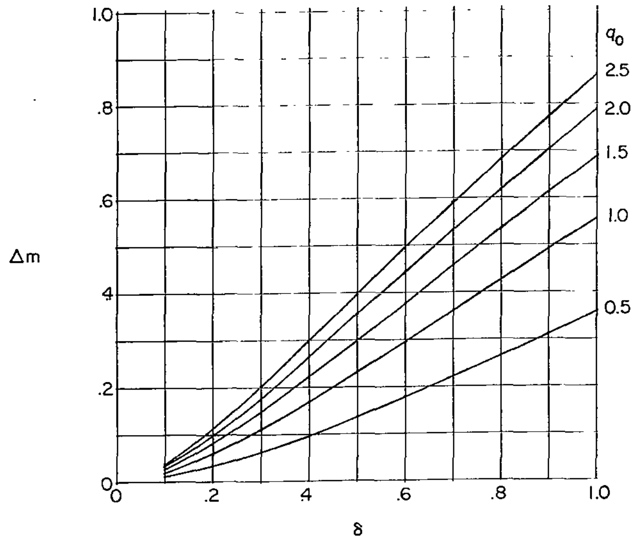

Figure 11.- Difference, as an incremental magnitude, introduced by use of equation (15) instead of equation (41).

This relationship differs from the expansion given by Mattig in reference 16 because terms on the order of δ^3 were retained in the MacLaurin expansion of the argument of the logarithm in equation (15). The terms of the order of δ^3 contribute the term $0.2715\left[4q_o(q_o - 1)\right]$ to the δ^2 of the logarithmic expansion because the common factor $c\delta/H_o$ is taken out before the logarithm is expanded. Mattig did not retain the δ^3 terms in the MacLaurin series; hence the noted difference. Comparison with equation (47) shows again that the differences are coming from the δ^2 and higher order terms. Since equation (48) includes the effects of curvature and density but not the cosmical constant, the differences are due to the neglect of density in equation (47) and the neglect of Λ in equation (48). There are no $\Lambda = 0$ models in the zero-density approximation that can be used for the analysis of data as these models occur for $q_o = 0$ and are not admissible. (See table II.) These results show the difference that is introduced when Λ is assumed to be zero and the present indications are that Λ is not zero in the observable universe.

The redshift-magnitude relation (eq. (41)) was used in a computing program to determine H_o and q_o from observational data. In this computing program the method of differential corrections (ref. 18) was used in a least-squares sense, that is, the sum of the squares of the residuals was minimized. The residuals were the differences between the corrected observed magnitudes and the computed magnitudes. The program looked for those values of H_o and q_o that minimized the sum of the squares of the residuals.

No limitation was placed on the sign and magnitude of q_o. In the case of H_o, because the observable universe is expanding and H_o must be positive, a stop was put in the computing program that would discontinue the computation if H_o became negative. The data used for this computation are presented in table V. Each data group was run with $\Delta M_o \neq 0$ and $\Delta M_o = 0$. When $\Delta M_o \neq 0$, it was set equal to $\dot{M}_o\delta/H_o$, $\dot{M}_o = 0.3$ magnitude per 10^9 years. (See ref. 19.) The δ/H_o part of the ΔM_o term is the first approximation to $t_o - t$. Equation (44) shows that $t_o - t$ is a function of H_o; consequently, when ΔM_o has a finite value, its partial derivative $\partial(\Delta M_o)/\partial H_o$ was included in the least-squares computation. The ΔM_o term accounts for evolutionary effects in galaxies and was included in this study in order to determine whether it has a significant effect on q_o and H_o. The simple first term of the expansion for $t_o - t$ (eq. (14)) was used instead of equation (45) because it was felt that in the present methods for determining $\dot{M}_o$, the errors present did not warrant the more exact formulation of $t_o - t$ and the approximate form of ΔM_o is more flexible and easier to use in the least-squares computation.

The data used in the least-squares computation are presented in table V. There are six groups of data presented and with the cited references this table is self-explanatory except for data group VI. Data group VI is a selected set of points from data groups I to III. These points were selected so that m_c always decreased, in the sense of magnitude, with increasing δ. The initial results of the least-squares computing program using these data are presented in table VI. These results were included in this paper in order to show possible values of q_o and H_o when the zero-density model is used for the analysis of observational data.

Three sets of data are presented in this table: $\Delta M_o = 0$, evolutionary effects not present; $\Delta M_o \neq 0$, evolutionary effects present; and the last where evolutionary effects were taken into account only when the redshift was greater than 0.2. The results obtained with these data are far too inconsistent to draw any specific conclusions; however, there are some interesting indications. The first and most important is that the problem of evolution must be resolved since when evolutionary effects were present the q_o values changed sign for 50 percent or more of the data groups. Secondly, to cut off evolution at $\delta = 0.2$, or less, is probably not correct because of the effect produced by one point at $\delta = 0.202$ in data groups I and III. The general inconsistency of the result indicates that a larger number of more accurate data points are needed at all values of the redshift. The root mean square of the residuals, which is really root-mean-square magnitude error, is too large, even in the best case, to be able to pick an exact model of the universe based on equation (41). Lastly, q_o is much more sensitive to observational data than H_o. This sensitivity occurs because q_o affects the characteristics of the curve and H_o plays the role of a scale factor. The values of q_o when positive are in general much larger than those usually quoted in current literature. This discrepancy is thought to arise from the fact that, as far as can be determined, an equation like equation (15) has heretofore been used to determine q_o whereas in this case equation (41) which is a much better formulation of the redshift-magnitude relation was used.

Data group II always gave the smallest root mean square. Whether this is an indication that these data are good and the values of H_o and q_o determined from the data are the most likely or whether the low root mean square occurred because of the sparseness of the data, is a question that cannot now be answered with certainty. The value of q_o is very sensitive to the quantity of data. For instance, with $\Delta M_o = 0$, the deletion of the seventh cluster in data group I changed q_o from 2.24 to 1.79. To answer the propounded question, it appears that the low root mean square associated with the results of data group II can probably be attributed to its sparseness.

TABLE V.- DATA USED IN LEAST-SQUARES COMPUTATION

[Position is given for 1950.0]

Data group I (ref. 19)

Position	δ	m_c	m_r
Virgo	0.004	9.16	
0316 + 4121	.018	12.51	
1257 + 2812	.022	12.84	
1603 + 1755	.036	14.12	
2308 + 0720	.043	14.78	
2322 + 1425	.044	15.04	
1145 + 5559	.052	15.71	
0106 - 1536	.053	15.21	
1024 + 1039	.065	15.88	
1239 + 1852	.072	15.22	
1520 + 2754	.072	15.93	
0705 + 3506	.078	16.26	
1431 + 3146	.131	17.31	
1055 + 5702	.134	17.31	
0025 + 2223	.159	17.39	
0138 + 1840	.173	17.16	
0925 + 2044	.192	17.54	
0855 + 0321	.202	17.84	

Data group II (ref. 12)

Position	δ	m_c	m_r
Virgo	0.004	9.2	
1257 + 2812	.022	12.8	
1520 + 2754	.072	15.6	
1055 + 5702	.134	16.9	
0925 + 2044	.192	17.4	
0024 + 1654	.29	18.7	
1448 + 2617	.35	18.5	
1410 + 5224	.44*	19.3	

Data group III (ref. 20)

Position	δ	m_c	m_r
1257 + 2812	0.022		13.5
0106 - 1536	.053		15.00
1024 + 1039	.065		16.00
1520 + 2754	.072		15.6
0705 + 3506	.078		15.4
0348 + 0613	.085		17.7
1513 + 0433	.094		16.0
1431 + 3146	.131		17.0
1055 + 5702	.134		17.0
2253 + 2341	.143		17.1
1534 + 3749	.153		17.0
0025 + 2223	.159		17.7
0138 + 1840	.173		17.9
1309 - 0105	.175		17.6
1304 + 3110	.183		17.7
0925 + 2044	.192		17.7
1253 + 4422	.198		17.7
0855 + 0321	.202		17.7

Data group IV

Position	δ	m_c	m_r
Virgo	0.004	9.16	
0316 + 4121	.018	12.51	
1257 + 2812	.022	12.84	
1603 + 1755	.036	14.12	
2308 + 0720	.043	14.78	
2322 + 1425	.044	15.04	
1145 + 5559	.052	15.71	
0106 - 1536	.053	15.21	
1024 + 1039	.065	15.88	
1239 + 1852	.072	15.22	
1520 + 2754	.072	15.93	
0705 + 3506	.078	16.26	
1431 + 3146	.131	17.31	
1055 + 5702	.134	17.31	
0025 + 2223	.159	17.39	
0138 + 1840	.173	17.16	
0925 + 2044	.192	17.54	
0855 + 0321	.202	17.84	
0024 + 1654	.29	18.7	
1448 + 2617	.35	18.5	
1410 + 5224	.44	19.3	

Data group V

Position	δ	m_c	m_r
1257 + 2812	0.022		13.5
0106 - 1536	.053		15.00
1024 + 1039	.065		16.00
1520 + 2754	.072		15.6
0705 + 3506	.078		15.4
0348 + 0613	.085		17.7
1513 + 0433	.094		16.0
1431 + 3146	.131		17.0
1055 + 5702	.134		17.0
2253 + 2341	.143		17.1
1534 + 3749	.153		17.0
0025 + 2223	.159		17.7
0138 + 1840	.173		17.9
1309 - 0105	.175		17.6
1304 + 3110	.183		17.7
0925 + 2044	.192		17.7
1253 + 4422	.198		17.7
0855 + 0321	.202		17.7
0024 + 1654	.29	18.7	
1448 + 2617	.35	18.5	
1410 + 5224	.44	19.3	

Data group VI**

Position	δ	m_c	m_r
Virgo	0.004	9.16	
0316 + 4121	.018	12.51	
1257 + 2812	.022	12.84	
1603 + 1755	.036	14.12	
2308 + 0720	.043	14.78	
2322 + 1425	.044	15.04	
0106 - 1536	.053	15.21	
1024 + 1039	.065	15.88	
1520 + 2754	.072	15.6	
0705 + 3506	.078	16.26	
1431 + 3146	.131	17.31	
0025 + 2223	.159	17.39	
0925 + 2044	.192	17.54	
1253 + 4422	.198	17.69	
0855 + 0321	.202	17.84	
1448 + 2617	.35	18.5	
1410 + 5224	.44	19.3	
3C295	.46	19.73***	

*0.44 was substituted for 0.46 of the reference as this is value given by Baum (ref. 21) for this cluster.
**This group comprises selected list of clusters from data groups I to III.
***Added to original and based on Minkowski's observation (ref. 22) corrected to photographic magnitudes and corrected for redshift.

TABLE VI.- INITIAL RESULTS FROM LEAST-SQUARES COMPUTATION FOR H_o AND q_o

Data group	$\Delta M_o = 0$			$\Delta M_o \neq 0$			$\Delta M_o \neq 0$ (*)		
	H_o	q_o	Root mean square	H_o	q_o	Root mean square	H_o	q_o	Root mean square
I	4.0×10^{-18}	2.24	0.331	3.98×10^{-18}	-0.536	0.3218	4.092×10^{-18}	1.106	0.336
II	4.54	2.934	.185	4.65	-.816	.171	5.012	-.94	.227
III	3.25	9.1	.437	3.51	1.119	.4357	3.516	5.45	.442
IV	3.82	4.637	.337	3.94	-.244	.311	4.23	-.1013	.330
V	3.04	12.79	.4186	3.61	.551	.327	4.17	4.55	.443
VI	3.95	4.025	.2812	4.08	-.691	.14	4.40	-.6355	.2837

*In this case ΔM_o applied only to data with a redshift greater than 0.2.

Basically, the results show that more and better data are needed to take even the first steps in resolving the problem of the structure and evolution of the universe. In addition, it must be determined whether the redshift is actually a Doppler shift and whether evolution (that is, the time variation of the luminosity of a galaxy) is present and, if so, its magnitude per epoch.

CONCLUDING REMARKS

A review of the problem of the cosmical constant in the field equations of general relativity, based on work by McVittie, Einstein, and Weyl, indicated that the cosmical constant is a constant of integration and should be retained in the field equations. A brief review of uniform models of the universe showed that the curvature constant and cosmical constant were sufficient to determine a model of the universe. These two parameters are functions of the density parameter, acceleration parameter, and Hubble parameter. These three parameters are determined from observational data.

A zero-density model of the universe was studied to determine its applicability to the analysis of observational data. It was found that under certain conditions the zero-density model universe would be very useful for the analysis of observational data. The zero-density model universe redshift-magnitude relation when used correctly showed smaller differences with respect to a finite-density-universe redshift-magnitude relation than other approaches to the redshift-magnitude relation that are currently in use.

Least-squares calculations for the Hubble parameter and acceleration parameter were made with the zero-density model universe redshift-magnitude relation, but the results were not conclusive because of the quality of the present magnitude data and the scarcity of data above redshifts of 0.2. When evolutionary terms were included in the calculation, the model changed; that is, the acceleration parameter changed sign. This result indicates a need for the better understanding of evolutionary effects in galaxies and an increase in the quantity and quality of data if progress is to be made on the problem of the structure and evolution of the universe.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., November 6, 1964.

APPENDIX

SYMBOLS USED IN ANALYSIS

The symbols used in the analysis are defined as follows:

c	speed of light in vacuum
D_l	luminosity distance
G	spur of Einstein-Ricci tensor
$G_{\mu\nu}$	Einstein-Ricci tensor
$g_{\mu\nu}$	metrical tensor
H_o	Hubble parameter, $\frac{\dot{R}_o}{R_o}$
$\dot{H}_o$	present rate of change of Hubble parameter
K	correction for redshift
k	space curvature constant
M_o	absolute magnitude of equivalent local source
ΔM_o	evolutionary correction to M_o to account for aging of galaxies as a function of travel time of light
m	apparent magnitude
m_c	corrected apparent magnitude
m_r	red apparent magnitude
$N(m)$	number of galaxies equal to or brighter than a given magnitude
n	number of galaxies per unit volume
p	pressure
Q	number of square degrees in celestial sphere
q_o	acceleration parameter, $-\frac{\ddot{R}_o}{R_o H_o^2}$

APPENDIX

R_o	present value of scale factor
$R(t)$	scale factor having dimensions of length and describing manner in which space unfolds with time
r,θ,ϕ	dimensionless coordinates of a point in metric subspace
$S(\omega)$	function defined by equation (22)
s	time along line element
$T_{\mu\nu}$	energy tensor
t	time
V_μ	velocity, $\frac{dx^\mu}{ds}$
x	coordinate of metric subspace
δ	redshift of spectral lines due to velocity of recession
δ_c	critical value of δ
ζ	dummy variable
κ	constant, $\frac{8\pi G}{c^2}$
Λ	cosmical constant
ξ,ω	metric subspace coordinates
ρ	density
ρ_o	present value of density
σ_o	density parameter, $\frac{4\pi G}{3H_o^2}\rho_o$

Dots over symbols denote derivatives with respect to time. Subscript o denotes present epoch, that is, present time.

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